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# Structure of Weak Descartes Systems

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A weak Descartes system is a basis of functions such that every ordered subset is a weak Tchebycheff system, the canonical example being the usual spline basis involving truncated power functions. By examining the intervals of degeneracy for a WD-system, we show that it is possible to produce a new basis that has a simple and convenient structure similar to the spline basis. 

In this paper we will apply results on degeneracy in WT-spaces, some of which were developed in [7], in order to investigate the structure of elements in a weak Descartes system.

DEFINITION 1. Let $u_0, ..., u_n$ be real-valued functions defined on a compact interval $[a, b]$. $\{u_0, ..., u_n\}$ is called a *weak Descartes* (WD) *system* if $\{u_{i_1}, ..., u_{i_k}\}$ forms a WT-system for all $0 \leqslant i_1 < \cdots < i_k \leqslant n$. If each of these subsystems is a T-system then $\{u_0, ..., u_n\}$ is called a *Descartes* (D) *system.*

We recall that $\{u_0, ..., u_n\}$ is a WT-system on $[a, b]$ if

$$\begin{pmatrix} u_0, ..., u_n \\ x_0, ..., x_n \end{pmatrix} = \det\{u_i(x_j)\}_{i,j=0}^{n} \geqslant 0$$

for all $a \leqslant x_0 < \cdots < x_n \leqslant b$. If these determinants are all positive then $\{u_0, ..., u_n\}$ is a T-*system*; it is a *complete* T-system if $\{u_0, ..., u_k\}$ is a T-system for $k = 0, ..., n$. It follows from Definition 1 that every element in a WD-system is nonnegative and every element in a D-system is positive.

D-systems and WD-systems have been investigated by Karlin and Studden [2] and by Krein and Nudel'man [3], among others. They were apparently introduced by Bernstein [1] and are so called because Descartes' rule of signs holds for elements in the linear span of a D-system (see [2]). According to this rule a function has at most the same number of zeros as its sequence of coefficients has sign changes. For WD-systems a similar result holds with "zeros" replaced by "sign changes."



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Bernstein [1] considered approximation by elements in the span of a D-system; this subject was taken up again by Smith [6]. Micchelli characterized best uniform approximation by elements in the span of a WD-system [5] (there the term "weak Markoff system" was used).

DEFINITION 2. Let $U$ be a linear space of functions defined on $[a, b]$. $U$ is called *degenerate* if a nontrivial element vanishes on an open subinterval of $[a, b]$; otherwise $U$ is called *nondegenerate*. A subinterval on which a nontrivial element vanishes is called a *degenerate interval* for $U$. If $U$ has a degenerate interval of the form $[a, \xi)$ we will say $U$ is *a-degenerate*; if $U$ has a degenerate interval of the form $(\xi, b]$ we will call $U$ *b-degenerate*.

A basis $\{u_0, ..., u_n\}$ will be referred to as degenerate when $\operatorname{span}\{u_0, ..., u_n\}$ is degenerate. Clearly, T-systems are nondegenerate; indeed, it is elementary that $\{u_0, ..., u_n\}$ is degenerate on an interval $I$ if and only if

$$\begin{pmatrix} u_0, ..., u_n \\ x_0, ..., x_n \end{pmatrix} = 0$$

for all $x_0 < \cdots < x_n$ in $I$.

Our first result concerns zeros of elements of a WD-system.

LEMMA 1. *Let $\{u_0, ..., u_n\}$ be a WD-system on $[a, b]$ and let $U = \operatorname{span}\{u_i\}_{i=0}^n$.*

(1) *If $U$ is not b-degenerate then $\mathscr{Z}(u_i; [a, b)) \subseteq \mathscr{Z}(u_{i+1}; [a, b))$ $(i = 0, ..., n-1)$, where $\mathscr{Z}(u; [a, b))$ denotes the zeros of $u$ in $[a, b)$.*

(2) *If $U$ is not a-degenerate then $\mathscr{Z}(u_i; (a, b]) \subseteq \mathscr{Z}(u_{i-1}; (a, b])$ $(i = 1, ..., n)$.*

(3) *If $U$ is neither a-degenerate nor b-degenerate then $\mathscr{Z}(u_i; (a, b)) = \mathscr{Z}(u_{i+1}; (a, b))$ $(i = 0, ..., n-1)$.*

*Proof.* (1) Assume that $u_i(x_0) = 0$ for some $x_0 \in [a, b)$ and some $0 \leqslant i \leqslant n-1$. As $U$ is not $b$-degenerate, there is an $x_1 \in (x_0, b]$ such that $u_i(x_1) > 0$. Since $\{u_i, u_{i+1}\}$ forms a WT-system we have

$$0 \leqslant \begin{pmatrix} u_i, u_{i+1} \\ x_0, x_1 \end{pmatrix} = -u_i(x_1) \cdot u_{i+1}(x_0);$$

hence, since $u_{i+1}$ is nonnegative, $u_{i+1}(x_0) = 0$.

Part (2) is proved similarly and part (3) is an immediate consequence of the first two parts. ▮

The statements in the following lemma appear in [7] and follow readily from [4, Lemma 1].

LEMMA 2. *If* $\{u_0, ..., u_{n-1}\}$ *is a T-system on* $[a, b]$ *and* $\{u_0, ..., u_{n-1}, u_n\}$ *is a WT-system, then* $\{u_0, ..., u_n\}$ *is either degenerate or else a T-system on* $[a, b]$. *In the former case there is an interval* $I \subset [a, b]$ *and a unique* $p \in \operatorname{span}\{u_0, ..., u_{n-1}\}$ *such that* $u_n - p \equiv 0$ *on* $I$, $u_n - p > 0$ *to the right of* $I$, *and* $(-1)^n(u_n - p) > 0$ *to the left of* $I$.

THEOREM 1. *Let* $\{u_0, ..., u_n\}$ *be a nondegenerate WD-system on* $[a, b]$. *If* $u_i(a) > 0$, $u_i(b) > 0$ $(i = 0, ..., n)$, *and at least one of the* $u_i$ *is positive on* $[a, b]$, *then* $\{u_0, ..., u_n\}$ *is a D-system.*

*Proof.* By Lemma 1 and the assumptions on $u_0, ..., u_n$, it follows that $u_i > 0$ $(i = 0, ..., n)$. For any $0 \leqslant i_1 < \cdots < i_k \leqslant n$ we may now apply Lemma 2 successively to $\{u_{i_1}, ..., u_{i_j}\}$ $(j = 2, ..., k)$ to show that $\{u_{i_1}, ..., u_{i_k}\}$ is a (complete) T-system on $[a, b]$ ($\{u_{i_1}\}$ is a one-dimensional T-system since $u_{i_1} > 0$). Hence $\{u_0, ..., u_n\}$ is a D-system. ∎

COROLLARY 1. *If* $\{u_0, ..., u_n\}$ *is a nondegenerate WD-system on* $[a, b]$ *with* $u_0 > 0$ *and* $u_i(a) > 0$ $(i = 1, ..., n)$ *then* $\{u_0, ..., u_n\}$ *is a D-system on* $[a, b]$.

*Proof.* From Lemma 2, $\{u_0, u_i\}$ is a T-system of dimension 2, from which it follows that $u_i/u_0$ is strictly increasing for $i = 1, ..., n$. In particular, $u_i(b) > 0$ $(i = 1, ..., n)$. Corollary 1 now follows from Theorem 1. ∎

THEOREM 2. *Let* $\{u_1, ..., u_n, v_1, ..., v_r\}$ *be a WD-system on* $[a, b]$ *such that* $\{u_1, ..., u_n\}$ *is a T-system, and assume that* $u_1(b) > 0$. *Then the following statements are valid*:

(1) $\{u_1, ..., u_n\}$ *is a complete T-system*; *if* $u_i(a) > 0$ $(i = 2, ..., n)$ *then* $\{u_1, ..., u_n\}$ *is a D-system. In any case,* $u_2, ..., u_n$ *are positive in* $(a, b]$.

(2) *If for some* $2 \leqslant i \leqslant n$, $u_i(a) = 0$ *then* $u_j(a) = 0$ *for all* $j = i, ..., n$ *and* $v_j(a) = 0$ *for all* $j = 1, ..., r$.

*Proof.* We observe that $u_1 > 0$ in $[a, b)$ since otherwise Lemma 1 implies that the $u_i$'s share a common zero, an impossibility for a T-system. Thus $u_1$ is positive in $[a, b]$ and we can use Lemma 2 (as in the proof of Theorem 1) to show that $\{u_1, ..., u_n\}$ is a complete T-system. If, in addition, $u_i(a) > 0$ $(i = 1, ..., n)$, then, by Corollary 1, $\{u_1, ..., u_n\}$ is a D-system. In any case, $u_2, ..., u_n$ must be positive in $(a, b]$ since $u_1 > 0$ implies that $\{1, u_i/u_1\}$ is a T-system for each $i = 2, ..., n$, so that $u_i/u_1$ is strictly increasing in $[a, b]$. This proves part (1). Part (2) follows from the proof of Lemma 1, since $u_i(b) > 0$ $(i = 1, ..., n)$. ∎

If $\{u_0, ..., u_n\}$ is a WD-system on $[a, b]$ with $u_0 > 0$ then, by Lemma 2, $\{u_0, u_1\}$ is either degenerate or else a T-system. Applying this analysis repeatedly, we see that there is a largest integer $k \geqslant 1$ such that $\{u_0, ..., u_{k-1}\}$ is a T-system (and thus a complete T-system). The classic example of such a system of functions is the basis

$$\{1, x, ..., x^{n-1}, (x-\xi_1)_+^{n-1}, ..., (x-\xi_r)_+^{n-1}\}$$

for the splines of degree $n-1$ on $[0, 1]$ with simple knots $\xi_1, ..., \xi_r$ (see [5]). Here $(x-\xi)_+^{n-1}$ is a truncated power function and equals $(x-\xi)^{n-1}$ for $x > \xi$ and is zero elsewhere. In order for this basis to be a WD-system it is crucial that $0 < \xi_1 < \cdots < \xi_r < 1$. Define $u_i(x) = x^{i-1}$ $(i = 1, ..., n)$ and $v_i(x) = (x - \xi_i)_+^{n-1}$ $(i = 1, ..., r)$. We observe that, for each $1 \leqslant i \leqslant r$, $\{u_1, ..., u_n, v_i\}$ is a degenerate WT-system, being degenerate both on $[0, \xi_i]$ and on $[\xi_i, 1]$. Moreover, as just noted, $\{\xi_i\}_{i=1}^r$ is an increasing sequence. Presently, we will demonstrate that these phenomena are intrinsically related to the weak Descartes nature of the spline basis.

DEFINITION 3. A linear space $U$ is said to be *maximally degenerate* on an interval $I$ if it is degenerate on $I$ but not on any interval strictly containing $I$.

Note that there may be many intervals on which $U$ is maximally degenerate. If $U$ comprises only continuous functions, then all maximal degenerate intervals of $U$ are closed.

LEMMA 3. *Let $\{u_1, ..., u_n, v_1, ..., v_r\}$ be a WT-system on $[a, b]$ such that $\{u_1, ..., u_n\}$ is a T-system and assume that, for some $1 \leqslant i \leqslant r$, $\{u_1, ..., u_n, v_i\}$ is maximally degenerate on an interval $I$ whose interior is $(\alpha, \beta)$. Then the following statements are valid*:

(1) *If $I$ exludes $b$ then $\{u_1, ..., u_n, v_j\}$ is degenerate on $I$ for each $j = i, ..., r$.*

(2) *If $I$ excludes $a$ then $\{u_1, ..., u_n, v_j\}$ is degenerate on $I$ for each $j = 1, ..., i$.*

(3) *If $I$ excludes both $a$ and $b$ then $\{u_1, ..., u_n, v_{i-1}\}$ is either maximally degenerate on $I$ or degenerate (at least) on $(\alpha, b]$ and $\{u_1, ..., u_n, v_{i+1}\}$ is either maximally degenerate on $I$ or degenerate on $[a, \beta)$.*

*Proof.* (1) Choose points $x_1 < \cdots < x_{n+1}$ in $I$. By Lemma 2, we may assume that $v_i \equiv 0$ on $I$ and $v_i(b) > 0$. Then for $i < j \leqslant r$

$$0 \leqslant \begin{pmatrix} u_1, ..., u_n, v_i, v_j \\ x_1, ..., x_{n+1}, b \end{pmatrix} = -v_i(b) \cdot \begin{pmatrix} u_1, ..., u_n, v_j \\ x_1, ..., x_{n+1} \end{pmatrix}.$$

Hence, since $\{u_1, ..., u_n, v_j\}$ is a WT-system, it follows that

$$\begin{pmatrix} u_1, ..., u_n, v_j \\ x_1, ..., x_{n+1} \end{pmatrix} = 0$$

for all $x_1 < \cdots < x_{n+1}$ in $I$ and so $\{u_1, ..., u_n, v_j\}$ is degenerate on $I$.

(2) As before, we may assume that $v_i \equiv 0$ on $I$ and, in this case, $(-1)^n v_i(a) > 0$. The proof now proceeds as in (1).

(3) By (2), $\{u_1, ..., u_n, v_{i-1}\}$ is degenerate on $I$. Suppose it is not maximally degenerate there and not degenerate on $(\alpha, b]$. Then, by (1), $\{u_1, ..., u_n, v_i\}$ must be degenerate on some interval properly containing $I$, a contradiction to the maximality of $I$. Similarly, it follows from (2) that $\{u_1, ..., u_n, v_{i+1}\}$ is either maximally degenerate on $I$ or ese degenerate on $[a, \beta)$. ∎

THEOREM 3. *Let* $\{u_1, ..., u_n, v_1, ..., v_r\}$ *be a WT-system of continuous functions on* $[a, b]$ *such that* $\{u_1, ..., u_n\}$ *is a T-system*[1] *and such that, for all* $1 \leqslant i_1 < \cdots < i_k \leqslant r$, $\{u_1, ..., u_n, v_{i_1}, ..., v_{i_k}\}$ *is a WT-system. Then there exist elements* $\tilde{v}_1, ..., \tilde{v}_r$ *such that* $\{u_1, ..., u_n, \tilde{v}_1, ..., \tilde{v}_r\}$ *is a basis for* $\operatorname{span}\{u_1, ..., u_n, v_1, ..., v_r\}$ *with the following properties:*

(1) $\{u_1, ..., u_n, \tilde{v}_{i_1}, ..., \tilde{v}_{i_k}\}$ *is a WT-system for all* $1 \leqslant i_1 < \cdots < i_k \leqslant r$.

(2) *The indices* $\{1, ..., r\}$ *may be partitioned into three segments (some possibly empty) such that*

(a) *for each* $j$ *in the first segment* $\tilde{v}_j \equiv 0$ *on an interval* $[\alpha_j, b]$ *and* $(-1)^n \tilde{v}_j > 0$ *in* $[a, \alpha_j)$;

(b) *either* $\{u_1, ..., u_n, \tilde{v}_j\}$ *is a T-system for every* $j$ *in the second segment, or else there is an interval* $[\alpha, \beta]$, $a < \alpha < \beta < b$, *on which every* $\tilde{v}_j$ *associated with the second segment vanishes,* $\tilde{v}_j > 0$ *in* $(\beta, b]$, *and* $(-1)^n \tilde{v}_j > 0$ *in* $[a, \alpha)$;

(c) *for each* $j$ *in the last segment* $\tilde{v}_j$ *vanishes on an interval* $[a, \beta_j]$ *and* $\tilde{v}_j > 0$ *in* $(\beta_j, b]$.

(3) *The sequences* $\{\alpha_j\}$ *and* $\{\beta_j\}$ *are nondecreasing and satisfy* $\max_j \alpha_j \leqslant \alpha \leqslant \beta \leqslant \min_j \beta_j$.

*Proof.* For every $1 \leqslant j \leqslant r$ for which $\{u_1, ..., u_n, v_j\}$ is nondegenerate (and hence a T-system) define $\tilde{v}_j = v_j$. Otherwise, $\{u_1, ..., u_n, v_j\}$ is maximally degenerate on some closed interval $[\alpha_j, \beta_j]$. Choosing such an interval, we define $\tilde{v}_j = v_j - p_j$, with $p_j \in \operatorname{span}\{u_1, ..., u_n\}$ as in Lemma 2. Then, for any $1 \leqslant i_1 < \cdots < i_k \leqslant r$ and $a \leqslant x_1 < \cdots < x_{n+k} \leqslant b$,

$$0 \leqslant \begin{pmatrix} u_1, ..., u_n\ v_{i_1}, ..., v_{i_k} \\ x_1, ..., x_{n+k} \end{pmatrix} = \begin{pmatrix} u_1, ..., u_n, \tilde{v}_{i_1}, ..., \tilde{v}_{i_k} \\ x_1, ..., x_{n+k} \end{pmatrix},$$

[1] Theorem 3 will normally be applied when $n$ is maximal in this respect.

which proves (1). By Lemma 3, we may select the intervals $[\alpha_j, \beta_j]$, and thus the corresponding elements $\tilde{v}_j$, so that all $\tilde{v}_j$ such that $\{u_1, ..., u_n, v_j\}$ is $b$-degenerate (briefly, "$b$-degenerate elements") come first, followed by nondegenerate elements or else a sequence of elements all maximally degenerate on the same interior interval, and finally any $a$-degenerate elements. The sign structure of these elements is dictated by Lemma 2. This proves part (2) of Theorem 3. By our choice of intervals $[\alpha_j, \beta_j]$ and from Lemma 3 it follows that, excluding those $\alpha_j$ equal to $a$ and those $\beta_j$ equal to $b$, the $\alpha_j$ and the $\beta_j$ form nondecreasing sequences with $\max_j \alpha_j \leqslant \alpha \leqslant \beta < \min_j \beta_j$. ▌

*Remarks.* (1) A result similar to Theorem 3 holds when continuity is not assumed. In that case, of course, the intervals of degeneracy need not be closed.

(2) It follows from Lemma 3 that if $\{\tilde{v}_1, ..., \tilde{v}_r\}$ contains any "nondegenerate elements" (as in Theorem 3 (2b)) then the $\tilde{v}_j$ are unique. For if $\tilde{v}_i$ is nondegenerate then, for $j \geqslant i$, $\tilde{v}_j$ may only be nondegenerate or $a$-degenerate, and for $j \leqslant i$ only nondegenerate or $b$-degenerate. Thus, by the uniqueness of the $p_j$ only one choice is possible for the $\tilde{v}_j$ $(j = 1, ..., r)$.

(3) If, in Theorem 3, $\{u_1, ..., u_{n-1}\}$ is a T-system as well, then any $v_j$, such that $\{u_1, ..., u_n, v_j\}$ is degenerate on an interval excluding $a$, must "involve" $u_n$ in the sense that $p_j = \sum_{i=1}^n a_i u_i$ with $a_n \neq 0$ (otherwise span$\{u_1, ..., u_n\}$ would be degenerate). Hence if $\{u_1, ..., u_{n-1}, v_1, ..., v_r\}$ satisfies the assumptions of Theorem 3, then the $\tilde{v}_i$ corresponding to this system are all either nondegenerate or $a$-degenerate. This indicates that if it is possible to "insert" a function $u_{n+1}$ into the system $\{u_1, ..., u_n, v_1, ..., v_r\}$ such that the new system satisfies the assumptions of Theorem 3, then the only possibility is that $\tilde{v}_1, ..., \tilde{v}_r$ are all nondegenerate or $a$-degenerate.

EXAMPLE 1. We return to the spline basis

$$\{1, x, ..., x^{n-1}, (x-\xi_1)_+^{n-1}, ..., (x-\xi_r)_+^{n-1}\}$$

for $x \in [0, 1]$ and $0 < \xi_1 < \cdots < \xi_r < 1$. As remarked earlier, this basis forms a WD-system on $[0, 1]$. We observe first that $\{1, x, ..., x^{n-1}\}$ is a complete T-system (although not a D-system) on $[0, 1]$ and that each of $x, x^2, ..., x^{n-1}$ vanishes solely at $x = 0$. This behavior is in accordance with Theorem 2. Further, the elements $(x-\xi_i)_+^{n-1}$ $(i = 1, ..., r)$ are each degenerate; that is, $\{1, x, ..., x^{n-1}, (x-\xi_i)_+^{n-1}\}$ is degenerate both on $[0, \xi_i]$ and on $[\xi_i, 1]$ (since $(x-\xi_i)^{n-1}$ is contained in span$\{1, x, ..., x^{n-1}\}$). Moreover, $\{\xi_i\}$ is an increasing sequence in keeping with Theorem 3. Finally, in the sense of Remark 3, each of the functions $(x-\xi_i)_+^{n-1}$ involves $x^{n-1}$, in keeping with the fact that they are degenerate on an interval excluding the left endpoint 0.

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